UNIT – 1 ELASTICITY

1.1 Introduction:

All bodies get deformed under the action of force. If the distance between any two points in a body remains invariable, the body is said to be a rigid body. In practice it is not possible to have a perfectly rigid body. The deformations are

(i) There may be change in length

(ii) There is a change of volume but no change in shape

(iii) There is a change in shape with no change in volume

The size and shape of the body will change on application of force. There is a tendency of body to recover its original size and shape on removal of this force.

1.2 Various Definitions:

Elasticity: The property of a material body to regain its original condition on the removal of deforming forces, is called elasticity.The nearest approach to perfectly elastic body is a quartz fibre.

Plasticity: The bodies which do not show any tendency to recover their original condition on the removal of deforming forces are called plasticity. Putty is considered to be the perfectly plastic body.

Load: The load is the combination of external forces acting on a body and its effect is to change the form or the dimensions of the body. Any kind of deforming force is known as Load.

When a body is subjected to a force it undergoes a change in size or shape or both. Elastic bodies offer appreciable resistance to the deforming forces. As a result, work has to be done to deform them. This amount of work is stored in body as elastic potential energy. When the deforming force is removed, its increased elastic potential energy produced a tendency in the body to restore the body to its original state of zero energy or stable equilibrium. This tendency is due to the internal forces which come into play by the deformation.

Stress: When a force is applied on a body, there will be relative displacement of the particles. Due to the property of elasticity, the particles tend to regain their original position. *The restoring or recovering force per unit area set up inside the bodyis called stress.*

- The stress is measured in terms of the load or the force applied per unit area. Hence its units are *dynes/cm*^{2} in CGS and *Newton/m*^{2}in MKS.
- It has a dimension **[$ML^{-1}T^{-2}$].** It is same as that of pressure.

There are two types of stress.

(1) Normal Stress: Restoring force per unit area *perpendicular to the surface*is called normal stress.

(2) Tangential or Shearing Stress: Restoring force per unit area *parallel to the surface* is called tangential or shearing stress.

Strain: The unit change produced in the dimensions of a body under a system of forces in equilibrium, is called strain. The strain being ratio. It is dimensionless quantity. It has no unit.

There are following three types of strains.

(1) Longitudinal or Linear Strain: It is defined as the change in length per unit original length of an object when it is deformed by an external force. There is no change in shape of the body.

The ratio of change in length to the original length is called longitudinal or linear strain.

$$i.e.\ \ Longitudinal\ or\ Linear\ Strain = \frac{Change\ in\ length\ (l)}{Original\ length\ (L)} \quad \ldots\ldots\ldots. \quad (1.1)$$

It is also called *Elongation strain* or *Tensile strain*.

(2) VolumeStrain: It is defined as change in volume per unit original volume, when an object is deformed by the external force. There is no change in shape of the body.

The ratio of change in volume to the original volume is called volume strain.

$$i.e.\ \ Volume\ Strain = \frac{Change\ in\ volume\ (v)}{Original\ volume\ (V)} \quad \ldots\ldots\ldots \quad (1.2)$$

(3) Shear strain: When the force applied parallel to the surface of the body then the change takes place only in the shape of the body. The corresponding strain is called shear strain.

The angular deformation produced by an external force is called shear strain. It is measured in radians.

Characteristics of a Perfectly Elastic Material

If a body is perfectly elastic then

- Strain is always same for a given stress.
- Strain vanishes completely when the deforming force is removed.
- For maintaining the strain, the stress is constant.

❖ Hooke's law

This fundamental law of elasticity was proposed by Robert Hooke in 1679and it states that "*Provided the strain is small, the stress is directly proportional to the strain*". In other words, *the ratio of stress to strain is a constant quantity for the given material* and it is called the *modulus of elasticity* or *coefficient of elasticity*.

$$Stress\ \propto Strain$$

$$\therefore\ Stress = E \times Strain$$

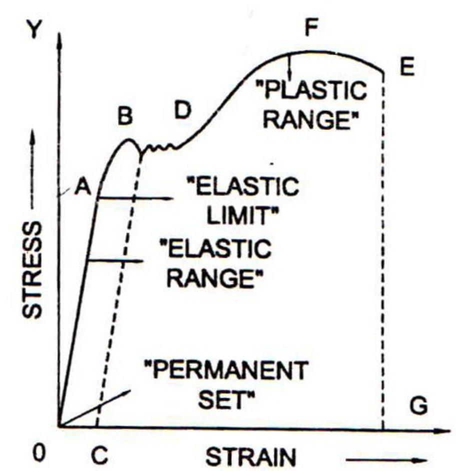
$$\therefore E = \frac{Stress}{Strain} \quad \ldots.\ldots\ldots \quad (1.3)$$

The units and dimensions of the modulus of elasticity are the same as that of stress.

❖ Elastic Limit:

When the stress is continually increased in the case of solid, a point is reached at which the strain increased more rapidly. The stress at which the linear relationship between stress and strain hold good is called elastic limit of the material.

❖ Stress-Strain Diagram:



The direct proportionality between stress and strain is found to be true only for small values of strain as shown in the figure 1.1

The portion OA of the curve is a straight line showing that the stress is directly proportional to strain. It shows that Hooke's law is strictly obeyed up to the value of stress corresponding to point A. This point is called ***Elastic limit***.

Beyond point A, the curve is not a straight line. In this region AB, the strain increases more rapidly than the stress and the behavior is partly elastic and partly plastic. If the object is unloaded at B, it does not come back to its original condition along path AO, but takes the dotted path BC. The object is said to

Fig: 1.1

have acquired ***permanent set***. And OC is called the ***residual strain***.

Beyond the point B, the length of the wire starts increasing without any increase in stress. Thus, wire begins to flow after point B and it continues up to D. The point B, at which the wire begins to flow is called ***yield point***.

Beyond the point F, the graph indicates that length of the wire increases, even if the wire is unloaded. The wire breaks ultimately at point E, called the ***breaking point*** of the wire. The portion of the graph between D and E is called the ***plastic region***.

1.3 Three types of elasticity:

There are three types of strain, therefore we have three types of elasticity.

(1) Linear elasticity called ***Young's modulus***, corresponding to *linear strain*.
(2) Elasticity of volume or ***Bulk modulus***, corresponding to *volume strain*.
(3) Elasticity of shape or shear modulus or ***Modulus of Rigidity***, corresponding to *shear strain*.

1.3.1 Young's Modulus:

When the deforming force is applied to the body only along a particular direction, the change per unit length in that direction is called *longitudinal, linear or elongation strain*.

The force applied per unit area of cross section is called *longitudinal or linear stress*.

The ratio of longitudinal stress to linear strain, within the elastic limit, is called Young's modulus. It is denoted by Y

$$Y = \frac{Longitudinal\ stress}{Linear\ strain} \quad \dots\dots\dots \quad (1.4)$$

Consider a wire of length L having area of cross section 'a', fixed at one end and loaded at the other end.

Suppose that a normal force F is applied to the free end of the wire and its length increase by l.

$$Longitudinal\ stress = \frac{F}{a} \quad and\ \ linear\ strain = \frac{l}{L}$$

$$Y = \frac{\frac{F}{a}}{\frac{l}{L}}\ \ or\ \ \ Y = \frac{FL}{al} \qquad \ldots.\ldots.\ldots \qquad (1.5)$$

Young's modulus can also be defined as the force applied to a wire of unit length and unit cross sectional area to produce the increase in length by unity.

The units of Young's modulus are **N/m²** in MKS and **dyne/cm²** in CGS system.

1.3.2 Bulk Modulus:

It is defined as the ratio of the normal stress to the volume strain**.** It is denoted by K. The bulk modulus is also known as the coefficient of cubical elasticity.

$$K = \frac{Normal\ stress}{Volume\ strain} \qquad \ldots\ldots.\ldots. \qquad (1.6)$$

Consider a cubic of volume V and surface area 'a'. Suppose that a force F which acts uniformly over the whole surface of the cubic, produces a decrease in its volume by v then,

$$Normal\ stress = \frac{F}{a}\ \ and\ \ volume\ strain\ = \frac{v}{V}$$

$$\therefore K = \frac{FV}{av}$$

Now, the pressure is$P = \frac{F}{a}$

$$\therefore K = \frac{-PV}{v}$$

If the volume increase on increasing the stress the bulk modulus given by

$$K = \frac{PV}{v} \qquad \ldots\ \ldots\ \ldots. \qquad (1.7)$$

The units of bulk modulus are Pa or N/m² in SI.

Compressibility: The reciprocal of the bulk modulus of a material is called compressibility i.e.**1/K.**

1.3.3 Modulus of Rigidity:

It is defined as the ratio of tangential stress to shear strain. It is also called shear modulus. It is denoted by η.

$$\eta = \frac{Tangential\ stress}{Shear\ strain} \qquad \ldots\ldots\ \ldots \qquad (1.7)$$

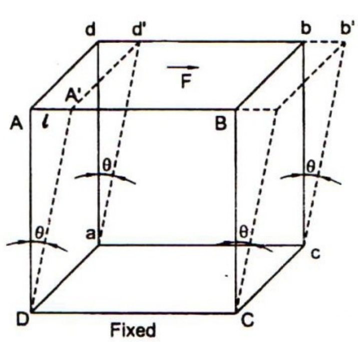


Fig: 1.2

Consider a rectangular block, whose lower face **aDCc** is fixed and the upper face **ABbd** is subjected to tangential force F as shown in Fig 1.2. Let 'a' be the area of the each face and AD = L be the perpendicular distance between them. The tangential force will displace the upper face of parallelepiped by a distance AA' = l. If$\angle ADA' = \theta$, then θ is the angle of shear.

$$Tangential\ stress = \frac{F}{a} \qquad and$$

$$Shear\ strain = Angle\ of\ shear = \theta$$

$$\eta = \frac{\frac{F}{a}}{\theta} = \frac{F}{a\theta}$$

For solids, angle of shear is very small, so in ΔADA'

$$\theta \approx tan\theta = \frac{AA'}{AD} = \frac{l}{L}$$

The distance '*l*' through which the upper face has been displaced is called lateral displacement.

$$\therefore \eta = \frac{F\ L}{a\ l} \qquad \dots\ \dots\ \dots\ (1.8)$$

1.4 Work done per unit volume in case of elongation strain:

In order to deform a body, work must be done by the applied force. The energy spent is stored in the body and is called the energy of strain. When the applied forces are removed, the stress disappears and the energy of strain appears as heat.

Let us consider the work done during the case of elongation strain or stretched of a wire. Consider a wire of length *l* and area of cross section '*a*' suspended from a rigid support. Suppose that a normal force '*F*' is applied at its free end and its length increases by *dl*.

The work done for a small displacement *dl* is given by

$$dW = F\ dl \qquad \dots\dots\dots\dots (1.9)$$

We know that,

$$Y = \frac{F/_a}{l/_L}$$

$$\therefore F = \frac{Y\ a\ l}{L}$$

Substituting this value of F in above equation (1.9), we get

$$dW = \frac{Y\ a\ l}{L}\ dl$$

Therefore, the total work done for the stretching a wire of length '*l*'given by,

$$W = \int_0^l dW$$

$$W = \int_0^l \frac{Yal}{L}\ dl$$

$$\therefore W = \frac{Ya}{L}\int_0^l l\ dl = \frac{Y\ a}{L}\left(\frac{l^2}{2}\right)$$

$$\therefore W = \frac{1}{2}\frac{Y\ a\ l}{L} \times\ l = \frac{1}{2}\ F\ \times l$$

$$\therefore Total\ work\ done\ W = \frac{1}{2}\ straching\ force \times stretch \qquad \dots\ \dots\ \dots\ (1.10)$$

This work done stored in form of potential energy.

Now, the volume of the wire = *a l*

$$\therefore Work\ done\ per\ unit\ volume = \frac{1}{2}\frac{F \times l}{a \times L} = \frac{1}{2}\left(\frac{F}{a}\right) \times \left(\frac{l}{L}\right) \quad \dots\ \dots\ \dots\ (1.11)$$

$$\therefore Work\ done\ per\ unit\ volume\ of\ the\ wire = \frac{1}{2} \times stress \times strain \quad \dots\ \dots\ \dots\ (1.12)$$

1.5 Deformation of cube:

When the force applied to a cube, the deformation produced in size, shape and volume. We can calculate the expression for corresponding elastic constants as follows.

1.5.1 Bulk Modulus:

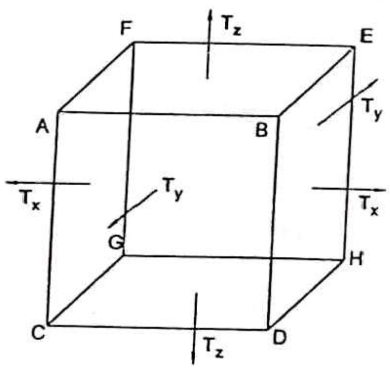


Fig: 1.3

Let us consider a unit cube ABDCGHEFA. Suppose force T_x, T_y and $T_z$are the force acting perpendicular to the faces BEHD and AFGC, ABDC and EFGH, ABEF and DHGC respectively, as shown in figure 1.3

Let 'α' be the increase per unit length per unit tension along the direction of the force and 'β' be the contraction produced per unit length per unit tension direction perpendicular to the force.

Due to the applied force, the elongations produce in the edges AB, BE and BD are $T_x\alpha$, $T_y\alpha$ and $T_z\alpha$ respectively. Similarly, the contraction produced in the perpendicular to these edges will be $T_x\beta$, $T_y\beta$ and $T_z\beta$.

The length of edges after elongation and contraction becomes,

$$AB = 1 + T_x\alpha - T_y\beta - T_z\beta$$
$$BE = 1 + T_y\alpha - T_x\beta - T_z\beta$$
$$BD = 1 + T_z\alpha - T_x\beta - T_y\beta$$

The volume of cube now becomes

$$V' = AB \times BE \times BD$$
$$V' = \left(1 + T_x\alpha - T_y\beta - T_z\beta\right) \times \left(1 + T_y\alpha - T_x\beta - T_z\beta\right) \times (1 + T_z\alpha - T_x\beta - T_y\beta)$$
$$V' = 1 + (\alpha - 2\beta)\,(T_x + T_y + T_z)$$

Neglecting squares and products of α and β.

In the case of bulk modulus, the force acting uniformly in all the directions,

Hence, $T_x = T_y = T_z = T$

$$\therefore V' = 1 + (\alpha - 2\beta)\,3T$$

The original volume of cube is unity; therefore increase in volume of the cube

$$v = V' - 1 = 1 + (\alpha - 2\beta)\,3T - 1$$
$$\therefore v = 3T(\alpha - 2\beta)$$

If pressure P is applied instead of tension T out words, the cube compressed and the volume decreased by the amount $3P(\alpha - 2\beta)$.

$$Volume\ strain\ = \frac{Change\ in\ volume}{Original\ volume} = \frac{3P(\alpha - 2\beta)}{1}$$

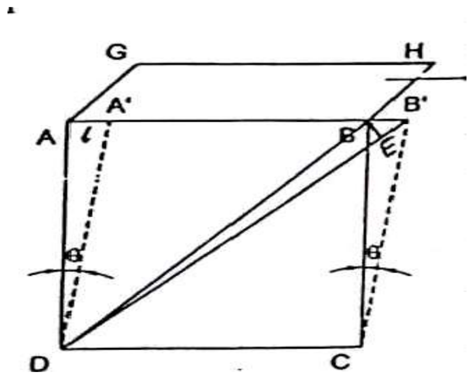
$$Bulk\ modulus\ K\ = \frac{Stress}{Volume\ strain} = \frac{P}{3P(\alpha - 2\beta)}$$

$$\therefore K\ = \frac{1}{3(\alpha - 2\beta)} \quad \dots\ \dots\ \dots\ (1.13)$$

$$Compressibility = \frac{1}{K} = \frac{1}{3(\propto - 2\beta)} \quad \dots\ \dots\ \dots\ (1.14)$$

1.5.2 Modulus of rigidity:

Consider a cube with an edge 'L'. Let shearing force $\vec{F}$ be applied on the top face ABHG of a cube, which produce shear by an angle θ and linear displacement 'l' as shown in fig 1.4



The face ABCD becomes A'B'CD.

$$Tensile\ stess = \frac{F}{area\ of\ face\ ABHG}$$

$$= \frac{F}{L^2} \quad = T$$

$$Shear\ strain\ = \frac{l}{L}$$

$$\therefore Modulus\ of\ rigidity\ \eta = \frac{Tensile\ stess}{Shear\ strain} = \frac{T}{\theta}$$

Fig: 1.4

A shearing stress along AB is equivalent to a tensile stress along DB and an equal compression stress along CA at right angles.

If α and β are the longitudinal and lateral strains per unit stress respectively.

Then extension along diagonal DB due to tensile stress = $DB\ T\ \alpha$ and, extension along diagonal DB due to compression stress along AC = $DB\ T\ \beta$.

Therefore, the total extension along DB = $DB\ T(\ \alpha + \beta)$

But, from above figure diagonal $DB = \sqrt{L^2 \times L^2}$

$$\therefore DB = \sqrt{2}L$$

Therefore, the total extension of diagonal $EB' = \sqrt{2}\ L\ T(\ \alpha + \beta)$ ... ... (1.15)

$$\text{In } \Delta \text{ BB'E, } Cos\ BB'E = \frac{EB'}{BB'}$$

$$\therefore EB' = BB'\ Cos\ \angle BB'E$$

$$But, BB' = l\ and\ \angle BB'E = 45^0$$

$$\therefore EB' = l\ Cos\ 45^0$$

$$\therefore EB' = \frac{l}{\sqrt{2}} \quad \text{... ... ... (1.16)}$$

Now, comparing equation(1.15) and (1.16), we get

$$\therefore \frac{l}{\sqrt{2}} = \sqrt{2}\ L\ T(\ \alpha + \beta)$$

$$\therefore \frac{L\ T}{l} = \frac{1}{2(\ \alpha +\ \beta)}$$

$$\therefore \frac{T}{\frac{l}{L}} = \frac{1}{2(\ \alpha +\ \beta)}$$

$$\therefore \frac{T}{\theta} = \frac{1}{2(\ \alpha +\ \beta)}$$

$$\therefore \eta\ = \frac{1}{2(\ \alpha +\ \beta)} \quad \text{... ... ... (1.17)}$$

1.5.3 Young's Modulus:

Let us consider unit tension applied on the edge of the unit cube, which produces the extension $'\alpha'$ linear stress =1 and linear strain = $\frac{\propto}{1}$= α.

$$Young's\ modulus = Y = \frac{1}{\alpha} \quad \text{... ... ... (1.18)}$$

1.6 Relation connecting the Elastic Constants:

We know that

$$K = \frac{1}{3(\alpha - 2\beta)}$$

and,

$$\eta = \frac{1}{2(\alpha + \beta)}$$

$$(\alpha - 2\beta) = \frac{1}{3K} \quad \ldots\ldots\ldots\ldots (1.19)$$

$$(\alpha + \beta) = \frac{1}{2\eta} \quad \ldots\ldots\ldots\ldots (1.20)$$

Subtracting (1) and (2)

$$3\beta = \frac{1}{2\eta} - \frac{1}{3K}$$

$$3\beta = \frac{3K - 2\eta}{6\eta K}$$

$$\beta = \frac{3K - 2\eta}{18\eta K} \quad \ldots\ldots\ldots\ldots (1.21)$$

Multiplying equation(1.20) by 2 and adding equations (1.19) and (1.20) we get,

$$3\alpha = \frac{1}{\eta} + \frac{1}{3K}$$

$$3\alpha = \frac{3K + \eta}{3K\eta}$$

$$\alpha = \frac{3K+\eta}{9K} \quad \ldots\ldots\ldots\ldots (1.22)$$

Form equation of young's modulus,

$$Y = \frac{1}{\alpha} \quad i.e\ \alpha = \frac{1}{Y} \quad \ldots\ldots\ldots\ldots (1.23)$$

Using equation(1.23) in (1.22)

$$\frac{1}{Y} = \frac{3K + \eta}{9K\eta}$$

$$\therefore \frac{9}{Y} = \frac{3K}{K\eta} + \frac{\eta}{K\eta}$$

$$\therefore \frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K} \quad \ldots\ldots\ldots\ldots (1.24)$$

The above equation gives the relation connecting the three elastic constants *Y, K* and η.

1.7 Poisson's Ratio:

When we stretch a wire, it becomes longer but thinner. The increase in its length is always accompanied with decrease in its cross section.

The strain produced along the direction of the applied force is called *primary or linear or tangential strain*(α) and strain produced at right angle to the applied force is called *secondary or lateral strain* (β).

Within the elastic limit, the lateral strain (β) is proportional to the linear strain(α) and the ratio between them is a constant, called Poisson's ratio(σ).

$$\sigma = \frac{Lateral\ strain}{linear\ strain} = \frac{\beta}{\alpha} \quad \ldots\ \ldots\ \ldots\ \ (1.25)$$

If the body under tension suffers no lateral strain then Poisson's ratio is zero.

1.7 Limiting values of 'σ' :

We know that,

$$3K(1-2\sigma) = 2\eta\,(1+\sigma)$$

Where, K and η are essentially positive quantities.

- Now ifσ is positive, then the RHS and hence LHS must be positive.
  This is true, if $1-2\sigma > 0$
  $$\therefore 2\sigma < 1$$
  $$\therefore \quad \sigma < \frac{1}{2}$$
  $$\therefore \sigma < 0.5 \qquad \ldots\ldots\ldots\ldots (1.26)$$
- Ifσ is negative, then the LHS and hence RHS must be positive.
  This is true, if $1+\sigma > 0$
  $$\therefore \quad \sigma > -1$$
  $$\therefore -1 < \sigma \qquad \ldots\ldots\ldots\ldots (1.27)$$
  Combining relation (1.26) and (1.27), we have
  $$-1 < \sigma < 0.5 \qquad \ldots\ldots\ldots\ldots (1.28)$$

Thus the limiting values of σ are -1 and 0.5. In actual practice, the value of σ lie between 0.2 to 0.4.

1.8 Determination of Poisson's Ratio for Rubber:

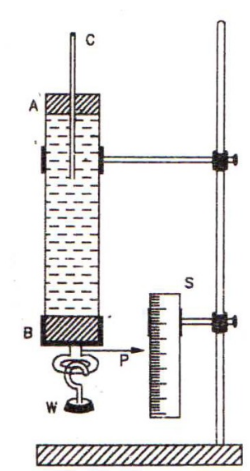


Fig: 1.5

To determine the value of σ for rubber, we take about a meter long tube AB and suspended vertically as shown in figure 1.5. Its two ends are property stopper with rubber corks and liquid glue. A glass tube C of half meter long and 1 cm in diameter is fitted vertically into the cork A through a suitable hole. A suitable weight W is then suspended from the lower end of the tube. This will increase the length and the internal volume of the tube.

It results in the fall of the level of meniscus in glass tube C. Both the increase in length (dL) and the decrease in the meniscus level (dh) are measured.

Let L, D and V be the original length, diameter and volume of the tube respectively. Then the area of cross-section of tube is

$$A = \pi r^2 = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{2} \ldots\ldots\ldots\ldots (1.29)$$

Differentiating above equation, we have

$$dA = \frac{\pi}{4} 2D\, dD = \frac{\pi D}{2} dD$$

$$\therefore\ dA = \frac{\pi D}{2} dD\ \frac{D}{2}\frac{2}{D}$$

$$\therefore dA = \frac{\pi D^2}{4} dD\ \frac{2}{D}$$

$$\therefore\ dA = \frac{2A\, dD}{D} \ldots\ldots\ldots\ldots (1.30)$$

Now, the increase in length of rubber tube dL and the increase in volume dV are accompanied with the decrease in area of cross section dA.

$$\text{Volume} = \text{area of cross section} \times \text{length}$$

$$\mathrm{V + dV = (A - dA)(L + dL)}$$

$$\therefore\ \mathrm{V + dV = A\,L + A\,dL - dA\,L - dA \cdot dL} \qquad \ldots\ldots\ldots\ldots (1.31)$$

Neglecting $\mathrm{dA \cdot dL}$ being very small.

We have,

$$\mathrm{V + dV = A\,L + A\,dL - dA\,L}$$

$$\therefore\ \mathrm{V + dV = V + A\,dL - dA\,L}$$

$$\therefore\ \mathrm{dV = A\,dL - dA\,L} \qquad \ldots\ldots\ldots\ldots (1.32)$$

Substituting the value of dA, we get,

$$\mathrm{dV = A\,dL - \frac{2A\,dD}{D}\,L} \qquad \ldots\ldots\ldots\ldots (1.33)$$

Dividing by dL on both sides,

$$\mathrm{\frac{dV}{dL} = A - \frac{2AL}{D}\frac{dD}{dL}}$$

$$\therefore\ \mathrm{\frac{2AL}{D}\frac{dD}{dL} = A - \frac{dV}{dL}}$$

$$\therefore\ \mathrm{\frac{dD}{dL} = \frac{D}{2AL}\left[A - \frac{dV}{dL}\right]}$$

$$\therefore\ \mathrm{\frac{dD}{dL} = \frac{D}{2L}\left[\frac{A}{A} - \frac{1}{A}\frac{dV}{dL}\right]}$$

$$\therefore\ \mathrm{\frac{dD}{dL} = \frac{D}{2L}\left[1 - \frac{1}{A}\frac{dV}{dL}\right]} \ldots\ldots\ldots\ldots (1.34)$$

Now Poisson’s ratio is given by

$$\sigma = \frac{Lateral\ Strain}{Liner\ Strain} = \frac{dD/D}{dL/L}$$

$$\therefore \sigma = \frac{L}{D} \times \frac{dD}{dL} \qquad \ldots\ldots\ldots\ldots (1.35)$$

Substituting the value of dD/dL from equation (1.34) in (1.35), we get

$$\sigma = \frac{L}{D} \times \mathrm{\frac{D}{2L}\left[1 - \frac{1}{A}\frac{dV}{dL}\right]}$$

$$\therefore \sigma = \frac{1}{2}\left[1 - \mathrm{\frac{1}{A}\frac{dV}{dL}}\right] \qquad \ldots\ldots\ldots\ldots (1.36)$$

If r be the internal radius of the tube, so that $\mathrm{A = \pi r^2}$

$$\therefore \sigma = \frac{1}{2}\left[1 - \mathrm{\frac{1}{\pi r^2}\frac{dV}{dL}}\right] \qquad \ldots\ldots\ldots\ldots (1.37)$$

If ‘a’ be the internal radius of the capillary tube, we have $\mathrm{dV = \pi a^2 \cdot dh}$

$$\therefore \sigma = \frac{1}{2}\left[1 - \mathrm{\frac{1}{\pi r^2}\frac{\pi a^2 \cdot dh}{dL}}\right]$$

$$\therefore \sigma = \frac{1}{2}\left[1 - \frac{a^2}{r^2}\frac{dh}{dL}\right] \qquad \ldots\ldots\ldots\ldots (1.38)$$

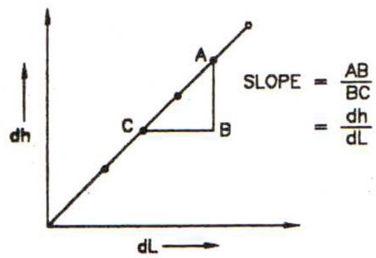


Fig: 1.6

The value of 'a' and 'r' are determined by a travelling microscope and a vernier caliper respectively and the average value of $^{dh}/_{dL}$ is obtained from the slope of the straight line graph by plotting a number of corresponding value of dh against the dL as shown in figure 1.6.

1.9 Twisting couple on a cylinder or wire:

When a cylinder or a wire is clamped at one end and twisted at the other end through an angle θ about its axis, then it will be under tension. Due to the elasticity of the material, a restoring couple is set up in it which is equal and opposite to the twisting couple.

Consider a cylindrical rod of length l radius r and coefficient of rigidity η. Its upper end is fixed and a couple is applied in a plane perpendicular to its length at lower end as shown in fig.1.7(a)

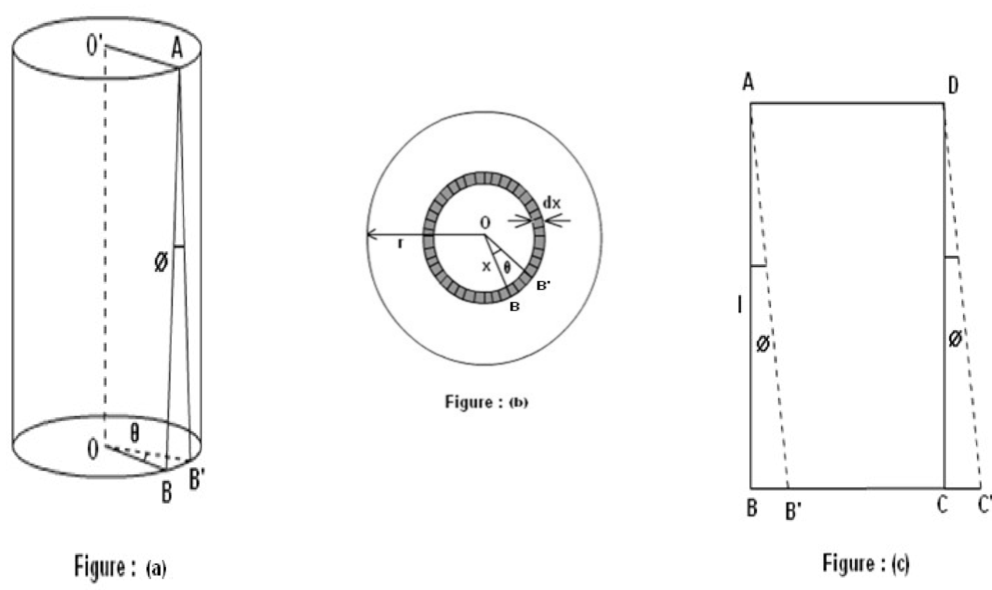


Fig:1.7

Consider a cylinder is consisting a large number of co-axial hollow cylinder. Now, consider a one hollow cylinder of radius x and radial thickness dx as shown in fig.1.7(b). Let θ is the twisting angle. The displacement is greatest at the rim and decreases as the center is approached where it becomes zero.

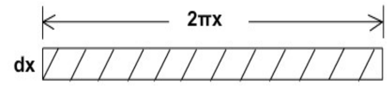
As shown in fig.1.7(a), Let AB be the line parallel to the axis OO' before twist produced and on twisted B shifts to B', then line AB become AB'.

Before twisting if hollow cylinder cut along AB and flatted out, it will form the rectangular ABCD as shown in fig.1.7(c). But if it will be cut after twisting it takes the shape of a parallelogram AB'C'D.

The angle of shear $\angle BAB' = \emptyset$

From fig.(c) $BB' = l\emptyset$

From fig.(b) $BB' = x\theta$



$$\therefore l\emptyset = x\theta$$

$$\therefore \emptyset = \frac{x\theta}{l} \ldots\ldots\ldots\ldots\ldots\ldots (1.39)$$

The modules of rigidity is

$$\eta = \frac{Shearingspress}{angleofshear} = \frac{F}{\phi}$$

$$\therefore F = \eta \cdot \emptyset = \frac{\eta x\theta}{l} \qquad \ldots\ \ldots\ \ldots \quad (1.40)$$

The surface area of this hollow cylinder $= 2\pi x dx$

$\therefore$ Total shearing force on this area$= 2\pi x dx \cdot \frac{\eta x\theta}{l}$ $\qquad \ldots\ \ldots\ \ldots$ (1.41)

$$= 2\pi\eta \frac{\theta}{l} x^2 dx \qquad \ldots\ \ldots.\ldots. \ (1.42)$$

Now, The moment of this force$= 2\pi\eta \frac{\theta}{l} x^2 dx \cdot x$ $\qquad \ldots\ \ldots\ \ldots$ (1.43)

$$= \frac{2\pi\eta\theta}{l} x^3 \cdot dx \qquad \ldots\ \ldots\ \ldots \quad (1.44)$$

Now, integrating between the limits x = 0 and x = r,

We have, total twisting couple on the cylinder

$$= \int_o^r \frac{2\pi\eta\theta}{l} x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \int_o^r x^3 dx$$

$$= \frac{2\pi\eta\theta}{l} \left[\frac{x^4}{4}\right]_o^r$$

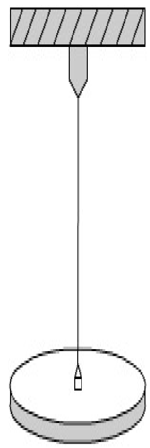
$$\therefore \text{Total twisting couple} = \frac{\pi\eta\theta r^4}{2l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.45)$$

Then, the twisting couple per unit twist $(\theta = 1)$ is

$$C = \frac{\pi\eta r^4}{2l} \qquad \ldots\ldots\ldots\ldots\ldots (1.46)$$

This twisting couple per unit twist is also called the ***torsional rigidity*** of the cylinder or wire.

1.10 Torsional Pendulum :



Definition: A heavy cylindrical rod or disc, suspended from the end of a torsion wire, whose upper end is fixed and a system is capable for oscillating in a horizontal plane is called *torsional pendulum*.

The rod or disc is turned in the horizontal plane to twist the wire. But the wire is torsion or elastic, hence it released and it execute torsional vibrations about the axis. Let θ be the twisting angle. Then the restoring couple set up in it is

$$C\theta = \frac{\pi\eta\theta r^4}{2l} \quad \ldots\ldots\ldots\ldots\ldots (1.47)$$

Fig: 1.8

This produces an angular acceleration $\frac{d\omega}{dt}$ in the rod or the disc. If I be the moment of inertia of the rod or the disc about the wire, we have

$$\therefore I\frac{d\omega}{dt} = -C\theta \qquad (\because\ \tau = I\alpha)$$

$$\therefore \frac{d\omega}{dt} = -\left(\frac{C}{I}\right)\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.48)$$

The negative sign indicates that restoring couple is in the opposite direction to the deflecting couple. The angular acceleration $\frac{d\omega}{dt}$ of the disc of rod is proportional to its angular displacement θ and therefore, its motion is simple harmonic.

Now, we can write

$$\therefore \frac{d\omega}{dt} + \left(\frac{C}{I}\right)\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.49)$$

$$\therefore \frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.50)$$

$$\therefore \frac{d^2\theta}{dt^2} + \mu\,\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.51)$$

where, $\mu = \frac{C}{I}$

Equation (1.51) is known as the equation of motion of torsional pendulum.

The periodic time of torsional pendulum is given by

$$t = 2\pi\sqrt{\frac{\text{displacement}}{\text{angular acceleration}}} = 2\pi\sqrt{\frac{\theta}{\left(\frac{C}{I}\right)\theta}}$$

$$t = 2\pi\sqrt{\frac{I}{C}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.52)$$

This is called the equation of time period for torsional pendulum.

Equation (1.52) can also be written as,

$$t = 2\pi\sqrt{\frac{\text{M.I of the disc or rod about the wire}}{\text{Restoring couple per unit twist of the wire}}} \quad \ldots\ldots\ldots (1.53)$$

From Equation (1.52), we can write,

$$t^2 = \frac{4\,\pi^2 I}{C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.54)$$

$$\therefore\ C = \frac{4\,\pi^2 I}{t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.55)$$

Using this equation we can determine the torsional couple per unit twist of the wire.

But, $C = \frac{\pi\eta r^4}{2l}$ and $I = \frac{MR^2}{2}$

Where, M is the mass and R is the radius of the disc.

Now, substituting the above values of C and I in equation (1.55).

We get,

$$\frac{\pi\eta r^4}{2l} = \frac{4\pi^2}{t^2}\frac{MR^2}{2}$$

$$\therefore \eta = \frac{4\pi^2 MR^2 l}{t^2 r^4} \ldots\ldots\ldots\ldots\ldots\ldots (1.56)$$

This is the expression for determination of modulus of rigidity η of a wire using torsional pendulum. The value of the radius of the wire should be measured accurately because in the equation it occurs in the fourth power of the radius of the experimental wire.

1.11 Determination of the coefficient of rigidity (η) for a write :

1.11.1 Statical Method : This method is based on the direct application of the expression for the twisting couplec = $\frac{\pi\eta r^4}{2l}$

- **Horizontal twisting apparatus for a rod :**

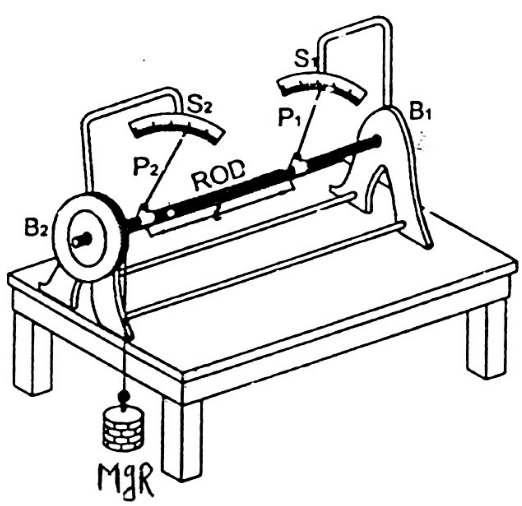


Fig: 1.9

The arrangement of the apparatus is shown in figure 1.9 A rod of 50 cm in length and about 0.25 cm in radius is fixed at one end with block B_1. A large pulley B_2 attached to the other end of the rod. A cord is wound round the pulley and mass M suspended at the other end. Hence couple acts on the rod and twisting produced in it. The Pointer $P_1$and P_2 are clamped on the rod a known distance 'l'. The twisting produced can be measured with the help of scale S_1 and S_2.

If R is the radius of the pulley, than couple acting on the rod isMgR, where M is mass suspended. This couple is balanced by the torsional couple due to rod and is equal to

$$= \frac{\pi\eta r^4 (\theta_2 - \theta_1)}{2l}$$

Where, r is the radius of rod, and θ_1 and θ_2 are the angles of twist produced at the two pointers.

Equating the two couples, we have

$$\frac{\pi\eta r^4 (\theta_2 - \theta_1)}{2l} = MgR$$

$$\therefore \ \eta = \frac{2MgRl}{\pi r^4 (\theta_2 - \theta_1)} \ \ldots\ldots\ldots \ (1.57)$$

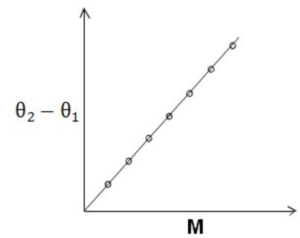


Fig: 1.10

The experiment is repeated with different masses and a graph is plotted between M and the twist $(\theta_2 - \theta_1)$. The slope of the straight line gives the mean of $\left(\frac{\theta_2 - \theta_1}{M}\right)$ which is used in the above expressions to find out η.

- **Draw backs of the statical method :**
  - (a) There being a pointer moving over the circular scale, an error is caused due to the eccentricity of the axis of the rod with respect to it.
  - (b) Since the force is applied through the pulley, a side pull is produced on the rod. This results in friction in the bearings which opposes the rod from twisting freely.

1.11.2 Dynamical Method:

In this method, a disc or rod, which is suspended from the wire and performing torsional vibrations about the wire and the time period of a body is determined. Maxwell derived a method using which we can easily determine the moment of inertia of a body without knowing the couple per unit twist.

- **Maxwell's Vibrating needle method :**

A hollow tube, open at both ends is suspended at the middle with the torsion wire whose modulus of rigidity is to be measured. It is suspended vertically from a support and a small piece of mirror attached to it, as shown in figures 1.11

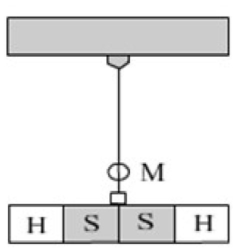


Figure : (a)

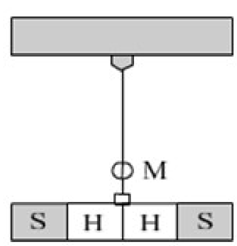


Figure : (b)

Fig: 1.11

As shown in fig.1.11(a), two hollow and two solid cylinder of equal length fitted into tube end to end. The solid cylinders are first into the inner position and hollow cylinders in the outer position as shown in fig.(a). The time period of a given system is given by

$$t_1 = 2\pi\sqrt{\frac{I_1}{C}} \qquad \dots\dots\dots\dots\dots\dots (1.58)$$

Where I_1 is the moment of inertia of the loaded tube and C is couple per unit twist of the wire.

Now, the position of hollow and solid cylinder are interchanged as shown in fig.1.11(b), Then, time period t_2 of second adjustment is given by

$$t_2 = 2\pi\sqrt{\frac{I_2}{C}} \qquad \dots\dots\dots\dots\dots\dots (1.59)$$

Where, I_2 is the moment of inertia of the tube in new position.

Squaring equation (1.58) and (1.59), we set

$$t_1^2 = \frac{4\pi^2 I_1}{C} \qquad \dots\dots\dots\dots\dots\dots (1.60)$$

And

$$t_2^2 = \frac{4\pi^2 I_2}{C} \qquad \dots\dots\dots\dots\dots\dots (1.61)$$

Subtracting (1.60) from (1.61), we have

$$t_2^2 - t_1^2 = \frac{4\pi^2}{C}(I_2 - I_1) \qquad \dots\dots\dots\dots\dots\dots (1.62)$$

Now, let m_1 be the mass of each hollow cylinder and m_2 be the mass of each solid cylinder. Let the length of tube be 2a. Hence, the length of each solid or hollow cylinder is $a/2$. Then the centers of mass of the inner and outer cylinders lie at distance $a/4$ and $3a/4$. Hence, in changing from first to second position an extra mass ($m_2 - m_1$) transferred from a distance $a/4$ to $3a/4$.

By the application of the principal of parallel axes, we have

$$I_1 = I_0 + 2I_S + 2m_2\left(\frac{a}{4}\right)^2 + 2I_H + 2m_1\left(\frac{3a}{4}\right)^2 \qquad \dots\dots\dots\dots\dots (1.63)$$

And,

$$I_2 = I_0 + 2I_H + 2m_1\left(\frac{a}{4}\right)^2 + 2I_S + 2m_2\left(\frac{3a}{4}\right)^2 \qquad \dots\dots\dots\dots\dots (1.64)$$

Where, I_o , I_S, and I_H are the moment of inertia of the hollow tube, solid cylinder and hollow cylinder about the axis of rotation respectively.

Now, subtracting equation (1.63) from (1.64), we get

$$I_2 - I_1 = 2(m_2 - m_1)\left(\frac{3a}{4}\right)^2 - 2(m_1 - m_2)\left(\frac{a}{4}\right)^2$$

$$\therefore I_2 - I_1 = 2(m_2 - m_1)\left[\left(\frac{3a}{4}\right)^2 - \left(\frac{a}{4}\right)^2\right]$$

$$\therefore I_2 - I_1 = 2(m_2 - m_1)\left[\frac{9a^2}{16} - \frac{a^2}{16}\right]$$

$$\therefore I_2 - I_1 = 2(m_2 - m_1)\frac{8a^2}{16}$$

$$\therefore I_2 - I_1 = (m_2 - m_1)a^2 \qquad \dots\dots\dots\dots\dots\dots(1.65)$$

Substituting this value of $I_2 - I_1$ in equation (1.62) we get

$$t_2^2 - t_1^2 = \frac{4\pi^2}{C}(m_2 - m_1)a^2$$

$$= \frac{4\pi^2}{\pi\eta r^4/_{2l}}(m_2 - m_1)a^2$$

$$= \frac{8l\pi^2 a^2}{\pi\eta r^4}(m_2 - m_1)$$

$$\therefore \eta = \frac{8\pi l a^2(m_2 - m_1)}{r^4(t_2^2 - t_1^2)} \qquad \dots\dots\dots\dots\dots\dots(1.66)$$

Thus, if we know l, a, m_1, m_2, t_1, t_2, and r, the modulus of rigidity (η) of the wire can be determined.

- **Advantage :**
  1. The total suspended mass from the wire remains same, hence value of C remains unchanged.
  2. There is no need to find the moment of inertia of the system, hence the question of uncertainty does not arise.

1.12 Bending of Beams :

A beam is a rod or a bar of uniform cross-section of a homogeneous, isotropic elastic material whose length is very large compared to its thickness. When a beam is fixed at one end and loaded at the other end as shown in figure 1.12(a) within the elastic limit, it will bend and couple produced inside it. The upper surface of the beam gets stretched and becomes a convex shape and lower surface gets compressed and becomes a concave form.

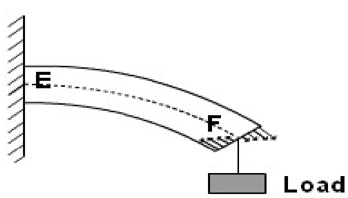


Figure : (a)

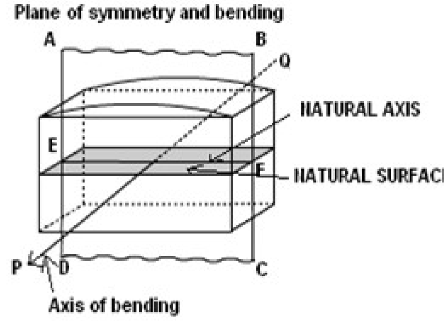


Figure : (B)

Fig: 1.12

All the longitudinal filaments in its upper half are extended and those in lower half are compressed. The extension is maximum in the uppermost filament and the compression is maximum in the lower most filaments. The amount of extension and compression decreases towards the axis of the beam. Thus filament EF neither extended nor compressed. This surface is called ***Neutral surface***. The plane in which all filaments are bent to form circular arcs is called the plane of bending. Thus in figure 1.12(b), plane ABCD is the plane of bending. The line perpendicular to the plane of bending is called the axis of bending. Thus, line EF is the neutral axis.

1.13 Bending Moment :

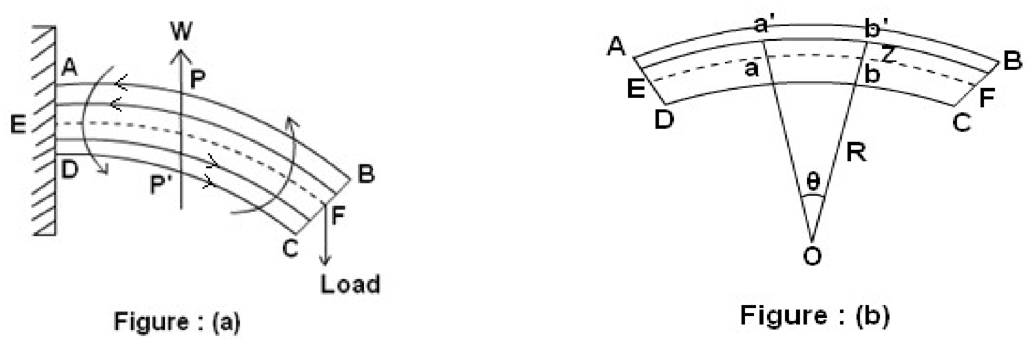


Fig: 1.13

Let a beam AB be fixed at A and loaded at B as shown in figure 1.13(a). EF is the neutral axis of the beam. Let us consider a section PBCP' cut by a plane PP' at right angles to its length. An equal and opposite reaction force W must be acting vertically upward direction along PP'. The beam bend or rotate in clock wise direction. The couple produced in the beam due to the load applied to the free end of the beam is called the **bending couple** and the moment of this couple is called the **bending moment**.

Let a small part of the beam bent in the form of a circular arc as shown in figure 1.13(b). This are subtending angle θ at 0. Let R be the radius of curvature of this part of the neutral axis. Let a'b' be an element at a distance Z from the neutral axis.

We know that , arc = Radius X angle subtended

$$\therefore\ a'b' = (R+Z)\cdot\theta$$

The original length $ab = R\cdot\theta$

$$\therefore \text{Increase in length} = a'b' - ab$$
$$= (R+Z)\cdot\theta - R\cdot\theta$$
$$= Z\cdot\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.67)$$

$$\text{Now, strain} = \frac{\text{Change in length}}{\text{Original length}}$$

$$= \frac{Z\cdot\theta}{R\cdot\theta} = \frac{Z}{R} \qquad \ldots\ldots\ldots\ldots\ (1.68)$$

Hence, the strain is proportional to the distance from the natural axis.
Now, consider a small area δa at a distance Z from the natural axis.

$$\text{Young's Modulus} \qquad Y = \frac{\text{Stress}}{\text{Strain}}$$

$$\therefore \text{Stress} = Y \text{ x Strain}$$

$$\therefore \frac{F}{\delta a} = Y \text{ x } \frac{Z}{R}$$

$$\therefore \textit{The force F on area } \delta a = Y \text{ x } \frac{Z}{R} \text{ x } \delta a \qquad \ldots\ldots\ldots\ldots (1.69)$$

Then, the moment of this force = Force x distance

$$= Y \text{ x } \frac{Z}{R} \text{ x } \delta a \text{ x } Z$$

$$= Y \text{ x } \frac{Z^2}{R} \text{ x } \delta a \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.70)$$

Then, the total moment of forces acting on all the filament is given by

$$\sum \frac{Y \cdot \delta a \cdot Z^2}{R} = \frac{Y}{R} \sum \delta a \cdot Z^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.71)$$

Here, $\sum \delta a \cdot Z^2$ is called the geometrical moment of inertia I_g of the section.

$$\therefore I_g = \sum \delta a \cdot Z^2 = ak^2 \qquad \ldots\ldots\ldots\ldots (1.72)$$

Where ‘a’ is the area of the surface and ‘k’ is the radius of gyration.

$$\therefore \text{The moment of forces} = \frac{Y}{R} \cdot ak^2$$

$$= \frac{Y}{R} \cdot Ig \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.73)$$

This is called the restoring couple or the bending moment of the beam.

∴ The bending moment M of beam is

$$M = \frac{Y}{R} \cdot Ig \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.74)$$

Here, the quantity $Y \cdot I_g = Y \cdot ak^2$ is called the flexural rigidity of the beam.

- **For rectangular cross section**, $a = b \text{ x } d$, and $k^2 = \frac{d^2}{12}$

$$\therefore I_g = ak^2 = bd \text{ x } \frac{d^2}{12} = \frac{bd^3}{12}$$

∴ The bending moment for rectangular cross section

$$M = \frac{Ybd^3}{12R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.75)$$

- **For circular cross section**, $a = \pi r^2$ and $k^2 = \frac{r^2}{4}$

$$\therefore \ I_g = ak^2 = \pi r^2 \text{ x } \frac{r^2}{4} = \frac{\pi r^4}{4}$$

$$\therefore \ M = \frac{Y \cdot \pi r^4}{4R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.76)$$

From above equation it is clear that, the bending moment M is directly proportional to the Young's modulus (Y) of the beam.

1.14 The cantilever :

Definition: A beam fixed horizontally at one end and loaded at the other end is called cantilever.

- **When the weight of the beam is ineffective :**

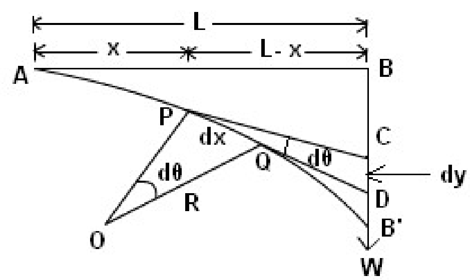


Fig: 1.14

Let AB be the natural axis of the cantilever of length L as shown in Figure 1.14. It is fixed at end and loaded at B with a weight W. Then the end B is depressed into the position B' and the natural axis takes up the position AB'. Consider a section P of the beam at a distance 'x' from the fixed end A.

The bending moment = W x PB′

$= W\,(L - x)$

Since the beam is in equilibrium.

∴ We can write

$$W\,(L - x) = \frac{YI_g}{R} = \frac{Yak^2}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.77)$$

Where, R is the radius of curvature.

Thus, for point Q at a small distance dx from P, we have

$$PQ = R \cdot d\theta$$

$$\therefore dx = R \cdot d\theta$$

$$\therefore R = \frac{dx}{d\theta}$$

∴ Equation (1.77) becomes,

$$W\,(L - x) = \frac{Y.ak^2.d\theta}{dx}$$

$$\therefore \ d\theta = \frac{W\,(L - x).dx}{Y.ak^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.78)$$

Now, the depression of Q below P is equal to CD or equal tody, then

$$dy = (L - x)d\theta$$

$$= \frac{(L-x)\cdot W(L-x)\cdot dx}{Yak^2}$$

$$dy = \frac{W\cdot(L-x)^2\cdot dx}{Yak^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.79)$$

Now, the total depression

$$y = \int_O^L dy = \int_O^L \frac{W\,(L-x)^2\,dx}{Y\cdot ak^2}$$

$$= \frac{W}{Y\cdot ak^2}\int_O^L (L^2 - 2Lx + X^2)dx$$

$$= \frac{W}{Y\cdot ak^2}\left[L^2x - 2L\frac{x^2}{2} + \frac{x^3}{3}\right]_O^L$$

$$= \frac{W}{Y\cdot ak^2}\left[L^3 - L^3 + \frac{L^3}{3}\right]$$

$$\therefore\ y = \frac{WL^3}{3Yak^2} = \frac{WL^3}{3YI_g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.80)$$

This is the expression of depression produced in the beam.

1.15 Depression of a beam supported at the ends :

(When the beam is loaded at the center):

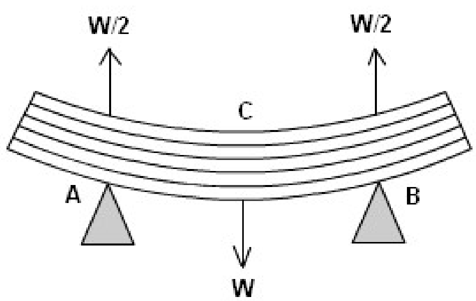


Fig: 1.15

Let a beam be supported on two knife edges at its two ends A and B, and let it be loaded in the middle at C with weight W as shown in figure 1.15.Since the middle part of the beam is horizontal, then the beam may be considered as equivalent to two inverted cantilevers, fixed at C and loaded at ends A and B by weight$^W/_2$.If L is the length of the beam AB, then the length of each cantileveris$^L/_2$.

Then the depression of C below A and B is given by

$$y = \frac{\left(^W/_2\right)\cdot\left(^L/_2\right)^3}{3\cdot Y\cdot ak^2}$$

$$\therefore y = \frac{WL^3}{48\,Y\cdot ak^2} = \frac{WL^3}{48\,Y\cdot I_g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.81)$$

- **If the beam is having a circular cross section,**

$$\text{then, } I_g = ak^2 = \frac{\pi r^4}{4}$$

$$y = \frac{WL^3}{48 \cdot Y} \cdot \frac{4}{\pi r^4} = \frac{WL^3}{12\,\pi r^4 \cdot Y} \qquad \dots\dots\dots\dots\dots\dots (1.82)$$

Where, r is the radius of cross section.

- **If the beam is having a rectangular cross section,**

$$\text{then, } I_g = ak^2 = \frac{bd^3}{12}$$

$$y = \frac{WL^3}{48 \cdot Y} \cdot \frac{12}{bd^3} = \frac{WL^3}{4Y \cdot bd^3} \qquad \dots\dots\dots\dots\dots\dots (1.83)$$

Solved Numerical

Ex-1.1

The Young's modulus of a metal is 2 X 10^{11} N/m^2 and its breaking stress is 1.078 X10^9N/m^2. Calculate the maximum amount of energy per unit volume which can be stored in the metal when stretched.

Sol: Here, Y = 2 X 10^{11} N/m^2

Maximum stress = 1.078 X 10^9 N/m^2

$$\text{Energy stored per unit volume} = \frac{1}{2}\text{ stress } \times \text{strain}$$

$$= \frac{1}{2}[1.078 \times 10^9] \times \frac{\text{stress}}{\text{Y}}$$

$$= \frac{1}{2} \times 1.078 \times 10^9 \times \frac{1.078 \times 10^9}{2 \times 10^{11}}$$

$$= 2.90 \times 10^6 \text{ J/m}^2$$

Ex-1.2

Find the work done in stretching a wire of 1 sq. mm cross section and 2 m long through 0.1 mm. Given Y = 2 X 10^{11} N/m^2.

Sol: As we know

$$Work\ done\ in\ stretching\ a\ wire = \frac{1}{2} \times Stretching\ force \times stretch$$

$$= \frac{1}{2} \times F \times l$$

$$= \frac{1}{2} \times \frac{Yal}{L} \times l$$

Here, Y = 2 X 10^{11} N/m^2

L = 2 m.

l = 0.1 mm = 10^{-4} m

a = 1 sq.mm = 10^{-6} m^2

$$\therefore Work\ done\ = \frac{1}{2} \times \frac{2 \times 10^{11} \times 10^{-6} \times 10^{-6} \times 10^{-4}}{2} = 5 \times 10^{-4}\ joules.$$

Ex: 1.3

The modulus of rigidity and Poisson's ratio of the material of a wire are 2.87 x 10^{10} N/m^2 and 0.379 respectively. Find the value of Young's modulus of the material of the wire.

Sol: Here, η = 2.87 x 10^{10} N/m^2 and σ = 0.379

We know that,

$$\eta = \frac{Y}{2\,(1+\sigma)}$$

$$\therefore Y = 2\eta\,(1+\sigma)$$

$$\therefore\ Y = 2 \times 2.87 \times 10^{10}(1 + 0.379)$$

$$\therefore Y = 7.915 \times 10^{10}\ N/m^2$$

Ex:1.4

A steel wire, 1 meter long and 1 mm square in cross section, supports a mass of 6kg. By how much does it stretch? (Give Y = 20 x 10^{10} N/m^{-2})

Sol: Here, L = 1 m

A = 1 sq. mm = 10^{-6} sq. m

m = 6 kg.

l = ?

The stretching force F = mg = 6 x 9.8 = 58.8 N

Young modulus$Y = \frac{F/a}{l/L}$

$$\therefore l = \frac{FL}{ay}$$

$$\therefore l = \frac{58.8\ \times 1}{10^{-6} \times 20 \times 10^{10}}$$

$$\therefore l = 2.94\ \times 10^{-4}\ m$$

∴ The increase in length of wire = 0.294 m

Ex: 1.5

A bronze bar 1.7 m long and 50 mm in diameter is subjected to a tensile stress of 70 Mega Newton / m^2. Calculate the extension produced in the bar and work done during the process. The value of Young's modulus for the material of the bar may be taken to be 85 x 10^9 N / m^2.

Sol:

Since Y = stress/strain, we have strain = stress / Y

And

Extension produced = strain x length

$$= \frac{stress}{Y} \times length = \frac{70 \times 10^6}{85 \times 10^9} \times 1.7 = 14 \times 10^{-4}\, m = 1.4\, mm$$

Now work done during stretch = 0.5 x stretching force x stretch

Here stretching force = tensile stress x area of cross section of the rod

$$= (70 \times 10^6) \times \pi \times 50^2 \times 10^{-6} / 4$$

$$\therefore work\ done = \frac{1}{2} \times \frac{(70 \times 10^6) \times \pi \times 50^2 \times 10^{-6}}{4} \times 14 \times 10^{-4}$$

$$= \frac{7 \times \pi \times 35}{8} = 96.23\, joules$$

Ex: 1.6

Calculate the twisting couple on a solid shaft of length 1.5 m and diameter 120 mm when it is twisted through an angle0.6^o. The coefficient of rigidity for the material of the shaft may be taken to be 93 x 10^9 N/m^2.

Sol:

$$C = \frac{\pi \eta r^2 \theta}{2\, l}$$

Here, θ = 0.6^o = $\frac{\pi}{180}$ x 0.6 radian

l = 1.5 m

r = $\frac{D}{2}$ = 60 mm = 0.06 m

η = 93 x 10^9 N/m^2

$$\therefore\ C = \frac{\pi \times 93 \times 10^9 \times (0.06)^4 \times 0.6\, \pi}{2 \times 1.5 \times 180}$$

$$\therefore\ C = 1.322 \times 10^4\ N\, m$$

Ex:1.7

A sphere of mass 0.8 kg and radius 0.03 m is suspended from a wire of length 1 m and radius 5 x 10^{-4} m. If the period of torsional oscillations of this system is 1.23 sec. Calculate the modulus of rigidity of the wire.

Sol:

Here,

$$t = 2\pi \sqrt{\frac{I}{C}}$$

But

$$I = \frac{2}{5} MR^2 \text{ for sphere}$$

and,

$$C = \frac{\pi \eta r^4}{2l}$$

$$\therefore t = 2\pi \sqrt{\frac{4MR^2 l}{5\pi \eta r^4}}$$

Squaring,

$$t^2 = \frac{16\pi MR^2 l}{5\eta r^4}$$

$$\therefore \eta = \frac{16\pi MR^2 l}{5t^2 r^4}$$

Here, M = 0.8 kg
R = 0.03 m
l = 1 m
r = 5 x 10^{-4} m
t = 1.23 sec.

$$\therefore \eta = \frac{16\pi \times 0.8 \times (0.03)^2 \times 1}{5 \times (1.23)^2 \times (5 \times 10^{-4})^4}$$

$$\therefore \eta = 7.654 \times 10^{10} N/m^2$$

Ex: 1.8

A cylindrical rod of diameter 14 mm rests on two knife – edges 0.8 m apart and a load of 1 kg is suspended from its mid-point. Neglecting the weight of the rod, calculate the depression of the mid-point if Y for its material be 2.04 x 10^{11} N / m^2.

Sol: We know that the depression y of the mid-point of a beam of circular cross-section, supported at the ends loaded in the middle is given by

$$y = \frac{WL^3}{12\ \pi r^4 \cdot Y}$$

Here, L = 0.8 m,
r = 0.014/2 = 0.007 m,
W = 1 x 9.81 N
And, Y = 2.04 x 10^{11} N/m^2

So that the depression of the mid-point of the beam is given by

$$y = \frac{9.81 \times (0.8)^3}{12 \times \pi \times (0.007)^4 \times 2.04 \times 10^{11}} = 0.000272\ m = 0.272\ mm$$

Ex: 1.9

A brass bar 1 cm square in cross section is supported on two knife edge 100 cm

apart. A load of 1 kg at the center of the bar depresses that point by 2.51 mm. What
is Young's modulus for brass?

Sol: We know that the depression of the mid – point of the bar is given by

$$y = \frac{WL^3}{48\,Y \cdot I_g}$$

Now, for a bar of rectangular cross – section, $I_g = b\,d^3 / 12$

Here, b = d = 1 cm, $\therefore b\,d^3 = 1$

W = mg= 1000 x 981 dynes

L = 100 cm

y = 2.51 mm = 0.251 cm

Therefore,

$$y = \frac{WL^3}{48\,Y \times b\,d^3/12} = \frac{WL^3}{4\,Y \times b\,d^3}$$

$$\therefore Y = \frac{WL^3}{4\,y \times b\,d^3} = \frac{981 \times 10^9}{4 \times 0.251} = 9.77 \times 10^{11} dynes/cm^2$$

Ex:1.10

A square metal bar of 2.51 cm side, 37.95 cm long, and weighing 826 gm is suspended by a wire 37.85 cm long and 0.0501 cm radius. It is observed to make 50 compete swings in 335.7 sec. What is the rigidity coefficient of the wire?

Sol:

Here, time period of the bar

t = 335.7 / 50 = 6.714 sec.

L=37.95 cm, B= 2.51 cm, M=826 gm, r=0.0501 cm and l=37.85 cm

Now, time period of a body executing a torsional vibration is given by

$$t = 2\pi\sqrt{I/C}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The moment of inertia of rectangular bar is $I = M\left(\frac{L^2+B^2}{12}\right)$

$$\therefore I = mass\left(\frac{length^2 + breadth^2}{12}\right) = 826 \times \frac{(37.95)^2 \times (2.51)^2}{12}$$

$$\therefore I = 826 \times \frac{1446.3}{12} = 99540\ gm\ cm^2$$

Substituting the value of periodic time t and moment of inertia I in equation (1), We have,

$$\therefore 6.714 = 2\pi\sqrt{99540 / C}$$

Squaring which, we have

$$C = 4\,\pi^2 \times 99540 / (6.714)^2$$

But, $C = \eta\,\pi\,r^4 / 2\,l$

$$\therefore \frac{\eta\, \pi\, r^4}{2\, l} = \frac{4\, \pi^2 \times 99540}{(6.714)^2}$$

$$\therefore \frac{\eta \times \pi (0.0501)^4}{2 \times 37.85} = \frac{4\, \pi^2 \times 99550}{(6.714)^2}$$

Then, coefficient of rigidity,

$$\eta = \frac{8\, \pi \times 99540 \times 37.85}{(0.0501)^4 \times (6.714)^2} = 3.357 \times 10^{11}\ dynes\ /\ cm^2$$

Exercise

(1) A wire 300 cm long and .625 sq. cm in cross – section is found to stretch 0.3 cm under a tension of 1200 kilograms. What is the Young's modulus of the material of the wire?(Ans.: 2.3 x 10^{12} dynes / sq.cm.)

(2) Calculate the work done in stretching a uniform metal wire of area of cross section 10^{-6} m^2 and length 1.5 m through 4 x 10^{-3} m. Given Y = 2 x 10^{11} N / m^2.
(Ans.: 1.066 Joule)

(3) Calculate the Poisson's ratio for the material given, Y = 12.25 x 10^{10} N / m^2 and η = 4.55 x 10^{10} N / m^2. (Ans.: 0.346)

(4) A uniform metal disc of diameter 0.1 m and mass 1.2 kg is fixed symmetrically to the lower end of a torsion wire of length 1 m and diameter 1.44 x 10^{-3} m whose upper end is fixed. The time period of torsional oscillations is 1.98 sec. Calculate the modulus of the rigidity of the material of the wire.
(Ans.: 3.579 x 10^{10} N / m^2)

(5) What couple must be applied to a wire, 1 meter long, 1 mm diameter, in order to twist one end of it through 90^0, the other end remaining fixed? The rigidity modulus is 2.8 x 10^{11}dynes cm^{-2}. (Ans. : 4.3 x 10^6 dynes cm)

(6) What couple must be applied to a wire 1 meter long and 2 mm in diameter in order totwist one of its ends through 45^0 when the other remains fixed.
Given η = 5 x 10^{11} dynes / cm^2. (Ans.: 6.1 x 10^5 dynes cm)

Question Bank

Multiple Choice Questions:

(1) Hooke's law essentially defines ________

(a)	Stress	(b)	Strain
(c)	Yield point	(d)	Elastic limit

(2) The dimensional formula of stress is _______

(a)	$[M^0L^1T^2]$	(b)	$[M^1L^{-1}T^{-2}]$
(c)	$[M^1L^{-1}T^{-2}]$	(d)	$[M^1L^{-1}T^{-1}]$

(3) The nearest approach to the perfectly elastic body is ________

(a)	Quarts fibre	(b)	Putty
(c)	Silver	(d)	Platinum

(4) ______ is the perfectly plastic material

(a) Quarts fibre (b) Putty

(c) Silver (d) Platinum

(5) The restoring force per unit area is called ________

(a) Stress (b) Strain

(c) Elasticity (d) Plasticity

(6) The change per unit dimension of the body is called _______

(a) Stress (b) Strain

(c) Elasticity (d) Plasticity

(7) The restoring force per unit area perpendicular to the surface is called ________

(a) Longitudinal Stress (b) Tangential Stress

(c) Normal Stress (d) Tensile Stress

(8) The restoring force per unit area parallel to the surface is called ________

(a) Longitudinal Stress (b) Lateral Stress

(c) Normal Stress (d) Tensile Stress

(9) Compressibility of a material is reciprocal of ____________

(a) Modulus of rigidity (b) Young Modulus

(c) Bulk Modulus (d) Coefficient of rigidity

(10) The work done per unit volume in stretching the wire is equal to ________

(a) Stress x strain (b) (1/2) stress x strain

(c) Stress / strain (d) Strain / stress

(11) The ratio of lateral strain to longitudinal strain is known as

(a) Young's modulus (b) Bulk modulus

(c) Poisson's ratio (d) Hooke's Law

(12) Theoretical value of Poisson's ratio lies between _________

(a) -1 and + 0.5 (b) - 1 and -2

(c) -0.5 and +1 (d) -1 and 0

(13) Which of the following relations is true?

(a) $3\alpha = \frac{3K - \eta}{3K\eta}$ (b) $3\alpha = \frac{3K + \eta}{3K\eta}$

(c) $\alpha = \frac{3K + \eta}{3K\eta}$ (d) $\alpha = \frac{3K - \eta}{3K\eta}$

(14) The relationship between Y, η and σ is ___________

(a) $Y = 2\eta (1 + \sigma)$ (b) $\eta = 2Y (1 + \sigma)$

(c) $\sigma = 2Y / (1 + \eta)$ (d) $Y = \eta (1 + \sigma)$

(15) The Poisson's ratio cannot have the value ________

(a) 0.7 (b) 0.2

(c) 0.1 (d) -0.52

(16) The Poisson's ratio can have the value _________

(a) 0.7 (b) -1.1

(c) 1.0 (d) 0.49

(17) Units of modulus of elasticity is ________

(a) dyne/cm (b) dyne/cm^2

(c) N/m (d) Dyne

(18) The relation between K, α and β is _________

(a) $K = \frac{1}{2(\alpha - 2\beta)}$ (b) $K = \frac{1}{3(\alpha - 2\beta)}$

(c) $K = \frac{1}{(\alpha - 2\beta)}$ (d) $K = \frac{1}{3(\alpha + 2\beta)}$

(19) In Bulk modulus, there is a change in the volume of the body but no change in ____

(a) Size (b) Shape

(c) Line (d) Angle

(20) Increase in the length of a wire is always accompanied by a decrease in ________

(a) Length (b) Breadth

(c) Cross section (d) Height

(21) The ratio of Longitudinal stress to linear strain is called _______

(a) Modulus of rigidity (b) Young Modulus

(c) Bulk Modulus (d) Coefficient of rigidity

(22) The ratio of Tensile stress to shear strain is called _______

(a) Modulus of rigidity (b) Young Modulus

(c) Bulk Modulus (d) Poisson's ratio

(23) The twisting couple per unit twist of a cylinder depends on _________

(a) Young's modulus (b) Bulk modulus

(c) Modulus of rigidity (d) Poisson's ratio

(24) If the material of a beam is _____, no bending should be produced.

(a) Homogenous (b) Isotropic

(c) Elastic (d) Plastic

(25) The unit of twisting couple is _________

(a) dynes/cm (b) N · m

(c) $N^2 \cdot m$ (d) $N \cdot m^2$

(26) On which of the followings the twisting couple per unit twist of a cylinder depends?

(a) Young's modulus (b) Bulk modulus

(c) Modulus of rigidity (d) Poisson's ratio

(27) The time period of the torsional vibrations(pendulum) is given by _________

(a) $t = 2\pi\sqrt{\frac{C}{I}}$ (b) $t = 2\pi\sqrt{\frac{I}{C}}$

(c) $t = 2\pi\sqrt{\frac{I}{Y}}$ (d) $t = 2\pi\sqrt{\frac{C}{K}}$

(28) In which part of cantilever the extension is maximum?

(a) Lowermost (b) Uppermost

(c) Middle (d) None of these

(29) The line of intersection of the plane of bending with the neutral surface perpendicular to is called the ________

(a) Neutral surface (b) Plane of bending

(c) Neutral axis (d) Axis of bending

(30) The material of a beam should not be _________

(a) Homogenous (b) Isotropic

(c) Elastic (d) Plastic

(31) The bending moment of a beam depends on only ________

(a) Young's modulus (b) Bulk modulus

(c) Modulus of rigidity (d) Poisson's ratio

(32) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the weight is doubled, the depression y will be

(a) y/2 (b) Y

(c) 2y (d) 4y

(33) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the length of the beam is doubled, the depression y will be ______

(a) y/8 (b) 2y

(c) 8y (d) 4y

(34) Let y is depression produced in the free end of cantilever when weight W is loaded at other end of the beam. If the length of the beam is reduced to L/2 and weight W/2, the depression y will be ________

(a) y/16 (b) 8y

(c) 16y (d) 4y

(35) The torsional rigidity of a cylinder is equal to ________

(a) $Y\frac{Ig}{R}$ (b) $Y\frac{R}{Ig}$

(c) Y x R×Ig (d) None of above

(36) The twisting couple per unit twist of wire or cylinder is also called __________

(a) Young Modulus (b) Modulus of rigidity

(c) Bulk Modulus (d) Torsional rigidity

(37) The twisting couple is equal and opposite to the _______

(a) Force (b) Pure shear

(c) Work (d) Restoring couple

(38) The periodic time of torsional pendulum depends on _________

(a) Young Modulus (b) Torsional rigidity

(c) Bulk Modulus (d) Amplitude of the oscillation

(39) The time period of a torsional pendulum is directly proportional to the square root of __________

(a) Distance (b) Vibrations

(c) Moment of inertia (d) Force

(40) The upper end of a wire of radius 4 mm and length 100 cm is clamped and its other end is twisted through an angle of 30^0. Then angle of shear is _______

(a) 12^0 (b) 0.12^0

(c) 1.2^0 (d) 0.012^0

(41) The depression produced in the free end of a cantilever is ________

(a) $y = \frac{W2L^3}{3\,Y\,I_g}$ (b) $y = \frac{3WL^3}{Y\,I_g}$

(c) $y = \frac{WL^3}{3\,Y\,I_g}$ (d) $y = \frac{LW^3}{3\,Y\,I_g}$

(42) Mathematical expression of flexural rigidity is ________

(a) Y^2ak (b) Ya^2k

(c) Ya^2k^2 (d) Yak^2

(43) The geometrical moment of inertia is given by ________

(a) $I_g = a^2k$ (b) $I_g = ak^2$

(c) $I_g = a^2/k$ (d) $I_g = k/a^2$

Short Questions:

(1) Explain (i) elasticity and (ii) plasticity.
(2) Explain: (i) load (ii) stress and (iii) strain.
(3) Explain: (i) Normal stress and (ii) Tangential stress.
(4) Explain: (i) linear strain (ii) volume strain and (iii) Shear strain.
(5) Define Young's modulus and Bulk Modulus.
(6) Define Modulus of rigidity
(7) Explain: Young Modulus and Modulus of rigidity
(8) State and explain Hook's law
(9) Define and explain Poisson's ratio
(10) Show that the value of Poisson's ratio lies between -1 and + 0.5.
(11) Describe stress-strain diagram for elongation of a wire.
(12) Derive relation between Young's modulus (Y), Modulus of rigidity (η) and Poisson's ratio (σ)
(13) Prove the relation, (i) $\frac{Y}{2\eta} = 1 + \sigma$ and (ii) $\frac{Y}{3K} = 1 - 2\sigma$.
(14) Define torsional pendulum and write the expression of its time period.
(15) Write the expression of torsional rigidity of wire.
(16) What is statical method?
(17) What is dynamical method?
(18) State the drawbacks of statical method of determination of modulus of rigidity.
(19) List the methods of determination of modulus of rigidity of a cylindrical rod or a wire.
(20) Discuss advantages of dynamical method for determination of modulus of rigidity.
(21) Define and explain bending moment.
(22) Define: cantilever and bending of beam.
(23) Explain the basic assumptions for the theory of bending.
(24) What is cantilever? Write expressions for depression of cantilever when the load is fixed at the center for rectangular and circular bar.
(25) Write name of the two experimental methods to determine (i) Young's modulus and (ii) modulus of rigidity.
(26) The upper end of a wire of radius 4 mm and length 100 cm is clamped and its other end is twisted through an angle of 30^0. Calculate the angle of shear.
(27) Define and discuss the terms (i) bending of a beam and (ii) Bending moment.
(28) Write only the equations for the depression of the mid-point of rectangular and cylindrical beams loaded at the centre and supported at ends.

Long Questions:

(1) Derive the formula for the work done per unit volume in stretching a wire.
(2) Prove that the work done per unit volume in stretching the wire is equal to $\frac{1}{2}(\text{stress} \times \text{strain})$.
(3) Explain three elastic constants in detail.
(4) Show the bulk modulus of elasticity is $K = \frac{1}{3(\alpha - 2\beta)}$

(5) Derive the expression of bulk modulus for deformation of a cube

(6) Derive the relation $\eta = \dfrac{1}{2(\alpha + \beta)}$ for deformation of cube

(7) Derive the expression of modulus of rigidity for deformation of a cube

(8) Discuss the case of deformation of a cube and derive the necessary expression for Three elastic constants and hence prove that $\frac{9}{Y} = \frac{3}{\eta} + \frac{1}{K}$

(9) Define Poisson's ratio and describe a method of determination. Derive the formula used.

(10) Define Poisson's ratio (σ) and derive formula of it to determine experimentally, i. e. $\sigma = \frac{1}{2}\left(1 - \frac{1}{A}\frac{dV}{dL}\right)$

(11) Derive the relation between three types of elastic module Y, K and η

(12) Define Poisson's ratio. Describe an experiment with necessary theory to determine the Poisson's ratio for rubber.

(13) Derive an expression for torsional rigidity of the cylinder or a rod of uniform circular section.

(14) Derive the equation for the couple per unit twist produced in a cylindrical wire or shaft with the help of necessary figures.

(15) Derive an expression for periodic time of a torsional pendulum $T = 2\,\pi\sqrt{\frac{I}{C}}$

(16) Define torsional pendulum and derive the equation for its time period.

(17) Discuss the torsional pendulum and derive its equation of motion.

(18) Explain the statical method of determination of modulus of rigidity and also mention its drawbacks.

(19) Describe dynamical method for determination of modulus of rigidity. Also discuss advantages of this method.

(20) Describe statical method (horizontal twisting apparatus for a rod) of determination of modulus of rigidity. Discuss drawbacks of the method.

(21) Describe dynamical method (Maxwell's vibrating needle method) of determination of modulus of rigidity.

(22) What is cantilever? Derive the equation for the depression produced in the free end of the cantilever if the weight of the beam is ineffective.

(23) What is bending moment? Derive the equation for the bending moment of beams having rectangular and circular cross-sections.

(24) Derive an expression for the depression of free loaded end neglecting weight of cantilever.

(25) Explain the concept of bending moment on the basis of theory of banding.

(26) What is bending moment? Show that the bending moment of a beam is $M = \frac{Y}{R} I_g$.

(27) Derive an expression for the depression of free loaded end neglecting weight of cantilever.

(28) Prove that the bending moment of beam is directly proportional to the Young Modulus.

(29) Obtain the formula for the depression of a beam supported at the ends and loaded at the centre.

(30) Derive an expression for depression of cantilever, when the load is fixed at the center. Also find the expression for rectangular and circular cross sections.

(31) Derive the expression bending of a tube supported at the 2 ends & loaded in the middle.

Answer key of MCQ:

(1)	(d)	(2)	(c)	(3)	(a)	(4)	(b)
(5)	(a)	(6)	(b)	(7)	(c)	(8)	(d)
(9)	(c)	(10)	(b)	(11)	(c)	(12)	(a)
(13)	(b)	(14)	(a)	(15)	(a)	(16)	(d)
(17)	(b)	(18)	(b)	(19)	(b)	(20)	(c)
(21)	(b)	(22)	(a)	(23)	(c)	(24)	(d)
(25)	(b)	(26)	(c)	(27)	(b)	(28)	(b)
(29)	(c)	(30)	(d)	(31)	(a)	(32)	(a)
(33)	(c)	(34)	(a)	(35)	(a)	(36)	(d)
(37)	(d)	(38)	(b)	(39)	(c)	(40)	(b)
(41)	(c)	(42)	(d)	(43)	(b)		